# MATH 112A Review: Geometric meaning of gradient and Linearization

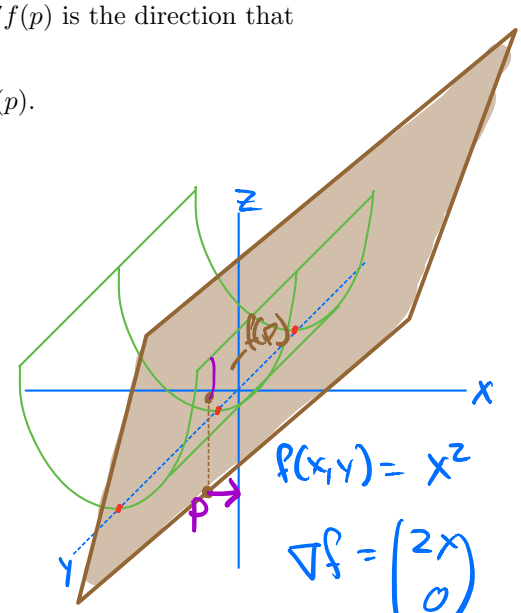
## Facts to Know:

Let $f : \mathbb{R}^2 \to \mathbb{R}$. we define the gradient $\nabla f$ of $f$ at the point $p$ to be the vector:

$$\nabla f(p) = \begin{pmatrix} f_x(p) \\ f_y(p) \end{pmatrix}$$

Geometrical interpretation:

- If $\nabla f(p)$ is a non-zero vector, then the vector $\nabla f(p)$ is the direction that the most quickly away from $p$.
- $||\nabla f(p)||$ is the rate of change in direction $\nabla f(p)$.




$f(x,y) = x^2$

$\nabla f = \begin{pmatrix} 2x \\ 0 \end{pmatrix}$

The linearization of $f$ at the point p is:

$$f(x,y) \approx L_f(x,y) = f(p) + \nabla f(p) \cdot \left( \begin{pmatrix} x \\ y \end{pmatrix} - p \right)$$

near p

## Examples: Let $f(x,y) = e^{xy^2}$.

1. What is the gradient of $f$ at the point $(1,2)$?

$$\nabla f(x,y) = \begin{pmatrix} y^2 e^{xy^2} \\ 2xy\, e^{xy^2} \end{pmatrix} \quad \nabla f(1,2) = \begin{pmatrix} 4e^4 \\ 4e^4 \end{pmatrix}$$

2. What direction increases the fastest from the point $(1,2)$ and what is the rate of change in the same direction?

The direction that increases the fastest from the point $(1,2)$ is $\begin{pmatrix} 4e^4 \\ 4e^4 \end{pmatrix}$.

The rate of change in that direction is

$$\left\| \begin{pmatrix} 4e^4 \\ 4e^4 \end{pmatrix} \right\| = \sqrt{(4e^4)^2 + (4e^4)^2} = \sqrt{2(4e^4)^2} = 4e^4 \cdot \sqrt{2}.$$

3. What is the linearization of $f$ at the point $(1, 2)$?

$$L_f(x,y) = e^4 + \begin{pmatrix} 4e^4 \\ 4e^4 \end{pmatrix} \cdot \left( \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right)$$

$$= e^4 + \begin{pmatrix} 4e^4 \\ 4e^4 \end{pmatrix} \cdot \begin{pmatrix} x-1 \\ y-2 \end{pmatrix}$$

$$= e^4 + 4e^4(x-1) + 4e^4(y-2)$$

$$= 4e^4 x + 4e^4 y + \left( e^4 + 4e^4(-1) + 4e^4(-2) \right)$$

$$A \cdot x + B \cdot y + C$$